An investigation of hypothetical variance-covariance matrix stress-testing

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Abstract Attempting to put meaningful numbers to portfolio risks is challenging. Conventional risk measures are considered often not to fully capture all risks inherent in a portfolio, particularly under difficult market conditions. Under such conditions stress-testing against artificial scenarios may help identify and quantify risks within a portfolio. Stress-tests also help reassure a portfolio or risk manager as to how a portfolio might respond to specific concerns.
This paper investigates an example of stress-testing a portfolio of conventional assets against market risks using artificial scenarios based around changes to the portfolio variance-covariance matrix. Hypothetical variance-covariance matrix stress-tests include making changes to correlations between assets to explore impacts on portfolio risks. Portfolio correlations, however, cannot be changed arbitrarily to reflect a risk manager's concerns without running the risk of implausible stressed returns and variance-covariance matrices that are not positive semi-definite. Different methods have been proposed in the literature to overcome this. This paper applies two such methods to a portfolio of four assets with the aim of illustrating the processes involved as well as drawing out differences in the approaches, enabling a discussion of their strengths and weaknesses.

Keywords: *portfolio, stress-testing, scenarios, market-risk, diversification, correlation*

INTRODUCTION

Portfolio stress-testing attempts to identify and quantify risks that are not well captured by more conventional measures, particularly regarding the impact on a portfolio of difficult market conditions. This paper investigates portfolio stress-testing using artificial scenarios known as hypothetical variance-covariance matrix stress-testing, aiming to provide straightforward examples that practitioners can follow and reproduce while drawing out differences in the approaches and discussing strengths and weaknesses.[1]

Portfolio managers may use stress-testing to explore portfolio downside risks under difficult ('stressed') conditions. Stress-testing cannot guarantee the identification of actual impacts on a portfolio of future events, but provides another tool in the risk manager's armoury. Stress-tests are designed to determine how a portfolio might respond to adverse developments, including portfolio allocation,[2] and detecting weak spots early, thus facilitating preventative action, typically focusing on key risks such as market risk, credit risk and liquidity risk.[3]

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Stress-testing covers a range of methodologies.[4] For current purposes it is sufficient to regard stress-tests as being either based on historical data ('historical stress-tests') or invented scenarios ('artificial stress-tests').[5]

One artificial, hypothetical stress-test adjusts the variance-covariance matrix of an asset portfolio to explore the impact that some anticipated change to asset relationships may have on overall portfolio risk.

As stress-testing tends to be an ad hoc practical activity rather than theoretically based,[6] a balance between art and science is required. Imagining dangerous scenarios is followed by efforts to examine their portfolio impacts. Determination of scenarios to be explored, and the magnitudes of anticipated changes in relationships between assets requires judgment, although the quantification of scenario impacts can be more scientific. Scenario selection depends on assumptions, which should broadly be regarded as 'unlikely but plausible'.[6]

HYPOTHETICAL STRESS-TESTING USING THE VARIANCE-COVARIANCE MATRIX

Volatility and value-at-risk (VaR) are often used to quantify risk; diversification through de-correlated assets reduces portfolio volatility and parametric VaR. Accepting the intuition that correlations often increase during market crashes,[7] to stress-test diversification one may increase correlations to quantify the impact this would have on portfolio risk. This encapsulates much of the logic behind variance-covariance matrix stress-testing; increased correlations are expected to increase portfolio volatility and VaR. The current paper focuses primarily on the correlation aspect.

For a multi-asset portfolio with volatility matrix[8] $\mathbf{v}$, the correlation matrix $\mathbf{R}$ yields the variance-covariance matrix $\mathbf{S} = \mathbf{vRv}$. The asset weight vector $\vec{w}$ gives the portfolio variance $\vec{w}^T \boldsymbol{S} \vec{w} = \sigma^2$, and portfolio parametric $VaR_{\%} = |-N\sigma\delta t^{1/2}|$, where N is the number of standard deviations for the confidence level required. Increasing both asset volatilities and correlations reflects some stressed scenario.

The stressed portfolio volatility can be used to obtain a parametric VaR; although common practice suggests applying a multiplier of four to the portfolio volatility[9] to obtain the stressed parametric VaR.

We cannot, however, modify the correlation matrix arbitrarily. Some combinations of correlations generate implausible stressed returns and variance-covariance matrices that are not positive semi-definite, meaning that negative variances can arise. Taking a correlation matrix from a stressed historical period avoids this, but makes the stress-test like a historical scenario, and may not explore correlations of primary concern. Alternatively, mathematical techniques can be used to construct the correlation matrix appropriately. Generally selected changes are made to some elements in the correlation matrix, while no view is expressed on the value of remaining matrix elements, which presumably should either be left unchanged, or else changed minimally. Rebonato and Jackel[2,10] observe that some methods have potentially undesirable side-effects such changing non-selected matrix elements. Methods may also require a pre-existing well-defined positive semi-definite matrix which is iteratively modified towards some target matrix. Clearly determination of an initial well-defined matrix may be a drawback, even if the potential slowness of iterative procedures may be mitigated by increasing computing power. Rebonato and Jackel propose a method that minimises an error measure to ensure that targeted elements are changed as desired while non-targeted elements are changed as little as possible. Although such an error measure is not formally used in the current investigation, the idea captures the desire to change non-selected matrix elements minimally. Higham[11] proposes minimising the weighted Frobenius norm as an error measure, the weights permitting an expression of the degree of confidence in different elements of the target correlation matrix. Turkay, Epperlein and Christofides[12] present the exact bounds for perturbing a single element in the matrix while keeping it positive definite and propose an iterative method to stress a group of correlations.

Two mathematical approaches are discussed here, which together with the detailed worked examples should prove helpful for a practitioner seeking to better understand the methods used. The side-effect of changes to non-selected correlations is illustrated as well as the desire to target selected correlations to different values while only minimally changing the others. The methods presented in the worked examples may be used by a practitioner in their own right, or else as an introduction to some of the other approaches discussed above.

If return histories on portfolio assets are available, the correlation matrix can be revised following Finger.[9,13] Correlations are adjusted by modifying selected return vectors with rescaling if original asset variances are to be unchanged. Not only are targeted correlations changed, but also other correlations in the same matrix rows and columns. Numpacharoen and Bunwong (N&B)[14] propose an alternative whereby the correlation matrix is adjusted directly. Cholesky decomposition ensures that a positive semi-definite correlation matrix is obtained, correlations are represented using trigonometrical functions and changes are made in correlative angles.[15]

PORTFOLIO DATA

The two approaches were explored using a four asset portfolio. Although small by asset manager standards, the limited number of assets is selected so that thoroughly worked examples can be presented. A correlation matrix was derived and selected correlations changed to explore their impacts. Given the manipulations to returns required by Finger's method,[13] a further simplification is made, with analysis conducted on just 12-monthly returns. This is inadequate for a meaningful formal risk analysis, but allows presentation of the results so that the reader can reproduce the examples shown.

Data covering the period to the end of August 2015 were used, which include a Chinese equity market correction. The assets were represented by indices and one ETF[16] with returns extracted from Yahoo Finance.[17]

(1) UK fixed interest: Lyxor ETF iBoxx GBP Gilts in GBP, denoted UK bonds, UKB or BD.
(2) UK equity: FTSE All-Share index, in GBP, denoted UK equity, or UK.
(3) US equity: S&P 500 index, in US$, denoted US equity, or US.
(4) Chinese equity: Shanghi Composite Index, in CNY, denoted CH equity, or CH.

Exchange rates to convert the US$ and CNY denominated indices into GBP were obtained from Oanda.[18] Use of publically available data sources such as Yahoo Finance and Oanda permits the reader to reproduce the analysis.[19]

The analysis period comprised monthly data from 30th April, 2014 to 31st August, 2015. This period is somewhat short with rather arbitrary start and end dates, but sufficient to demonstrate the methods used, and should be readily extensible to other periods and sampling frequencies.[20] Figure 1 shows

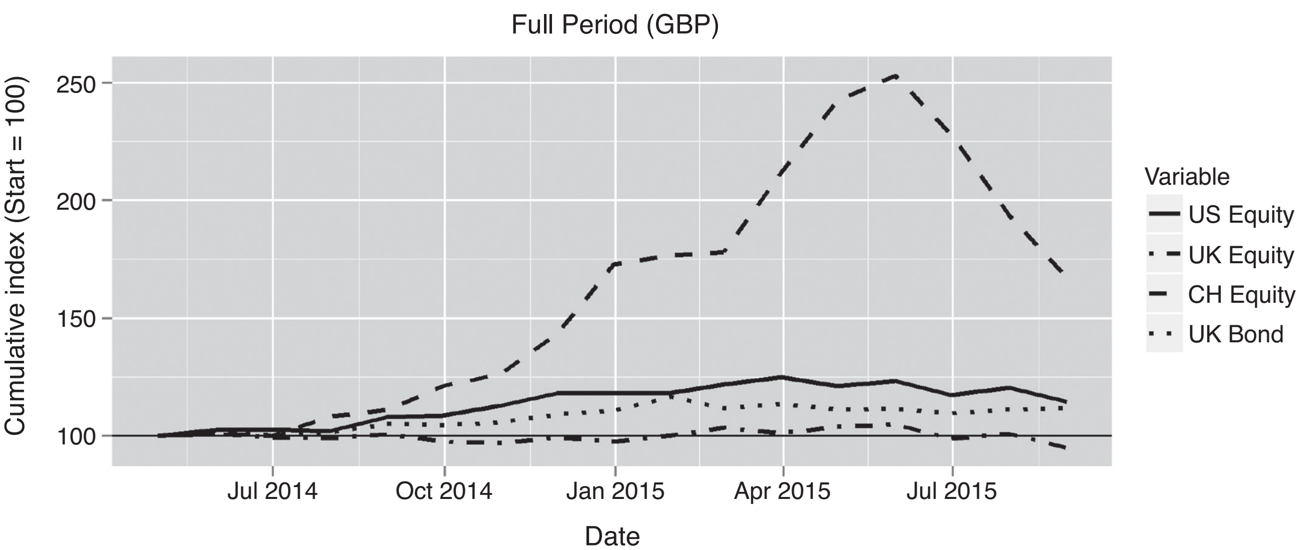

Figure 1: Cumulative index of the four asset returns over the full period analysed (30st April, 2014–31st August, 2015)

the cumulative index for the performances of the four assets over the period analysed. Chinese equity has generated significantly higher returns than other assets, although with a significant correction following June 2015. Observing the directions of movement in the index values from month to month, elements of decorrelation can be seen.[21]

To add realism, the full period data were divided into two overlapping 12-month sub-periods: 30th April, 2014 to 30th April, 2015, and 31st August, 2014 to 31st August, 2015. These two periods represent a period in the run-up to the Chinese equity market correction commencing June 2015, and a later period encompassing the correction. Figure 2 shows the cumulative indexed performances for these sub-periods.

The selection of these two periods was intended to be realistic from a risk practitioner perspective.

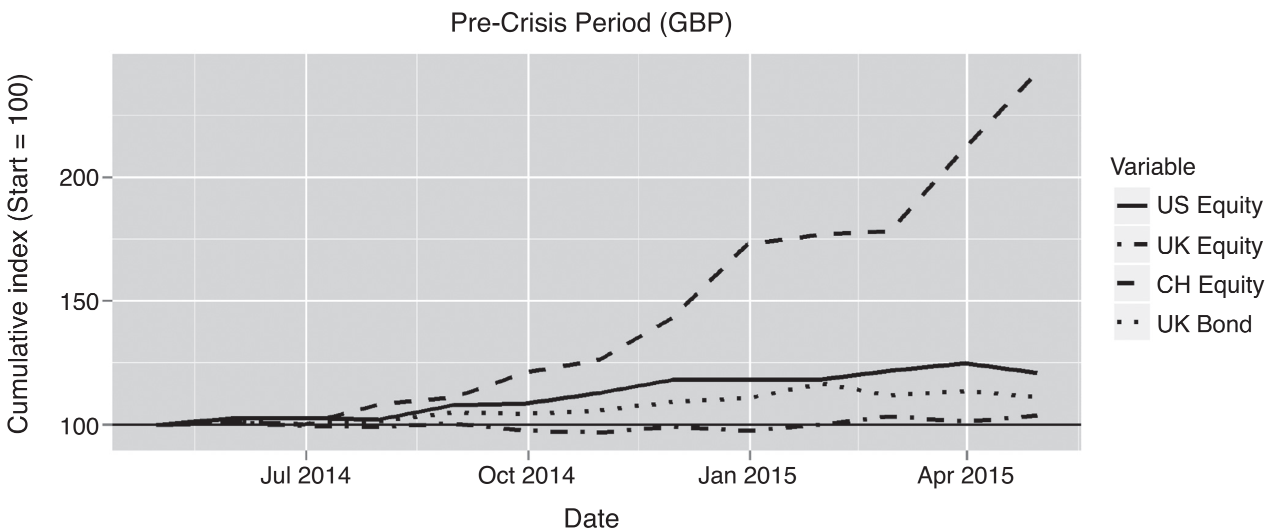

Figure 2a: Cumulative index of the four assets in the 12-month period to 30st April, 2015, representing performance in the run-up to the Chinese correction in June 2015

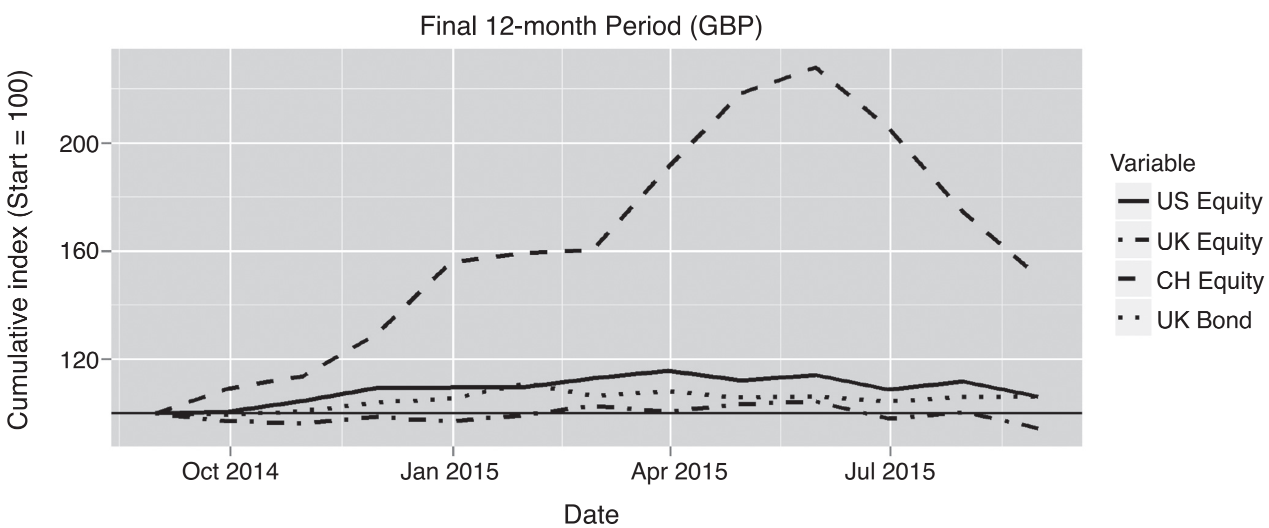

Figure 2b: Cumulative index of the four assets in the 12-month period to 31st August, 2015, representing performance over a period encompassing the Chinese correction in June 2015

While the periods analysed (one year) and sampling frequency (monthly) are unrealistic,[22] conducting a stress-test based on data prior to some crisis (or difficult market period) and how the results of the stress-tests compare with data observed during that period is highly pertinent. Also, as calculations of volatility and VaR estimates require data collection over some finite period to date, pre- and post-event stress-tests will be likely to involve periods with overlapping datasets.

The approach of this paper is to construct stress-tests based on the earlier sub-period and then, after illustrating the method in detail, compare the results with risk estimates for the later sub-period.

INITIAL PERIOD VAR ESTIMATE

Commence by calculating the parametric VaR of the portfolio over the initial sub-period from 30th April, 2014 to 30th April, 2015. Twelve monthly returns were calculated from the index values for each asset in percentage terms[23] (Table 1).

Annualised volatilities[24] populated volatility matrix **v**. The correlation matrix (with assets presented in the order listed above) was[25]

$$\boldsymbol{R} = \begin{bmatrix} 1 & -0.0017 & 0.2941 & 0.0611 \\ -0.0017 & 1 & 0.1553 & -0.2914 \\ 0.2941 & 0.1553 & 1 & -0.2319 \\ 0.0611 & -0.2914 & -0.2319 & 1 \end{bmatrix}.$$

For portfolio volatility, asset weights were required. Here, an essentially equal-weighted portfolio, with a lower weight in risker Chinese equities was selected, $\vec{w}^T = (0.3, 0.3, 0.3, 0.1)$.[26] The resulting portfolio variance, $\vec{w}^T \boldsymbol{S} \vec{w} = \sigma^2$, gave a portfolio volatility for the initial sub-period of 5.09 per cent p.a. and a 95 per cent monthly parametric VaR of 2.42 per cent.[27]

SCENARIO SELECTION

For variance-covariance matrix stress-testing, scenario selection includes deciding which correlations to adjust and values to use. A practitioner could use a correlation matrix for the portfolio from some interesting historical period. In this case it essentially becomes historical stress-testing. Additionally, using a historical correlation matrix may have drawbacks: data may not be available for the period of interest; although some correlations in the matrix may rise, others may fall making the stress-test less demanding; and the available historical record may not include periods that address the practitioner's primary concerns. Thus manual adjustment of the correlation matrix may be required.

The variance-covariance matrix stress-tests discussed here permit a practitioner to deliberately target specific correlation values, while other matrix values are adjusted to ensure that overall the correlation matrix retains necessary mathematical properties.

Judgment is required as to which correlations to adjust, and what values to use. Thereafter the techniques of Finger,[9,13] or N&B[14] ensure that variance-covariance matrices that are positive

Table 1: Percentage monthly returns for the initial sub-period, 30th April, 2014–30th April, 2015

Date	UK Bond	UK Equity	US Equity	CH Equity
31/05/2014	0.9017	0.9719	2.5667	0.947
30/06/2014	–0.5142	–1.4999	0.1485	–1.1093
31/07/2014	1.0592	–0.4047	–0.8872	8.0446
31/08/2014	3.5049	1.5038	5.8208	3.1283
30/09/2014	–0.6643	–2.9017	0.6385	8.7887
31/10/2014	1.4272	–0.8622	3.882	4.3254
30/11/2014	3.2403	2.5649	4.7312	13.2434
31/12/2014	1.3474	–1.6859	0.3135	21.264
31/01/2015	5.3792	2.5213	–0.0789	2.0892
28/02/2015	–4.2004	3.3809	3.0105	0.7217
31/03/2015	1.5985	–2.1548	2.1708	18.3803
30/04/2015	–2.0978	2.6335	–2.7236	14.743

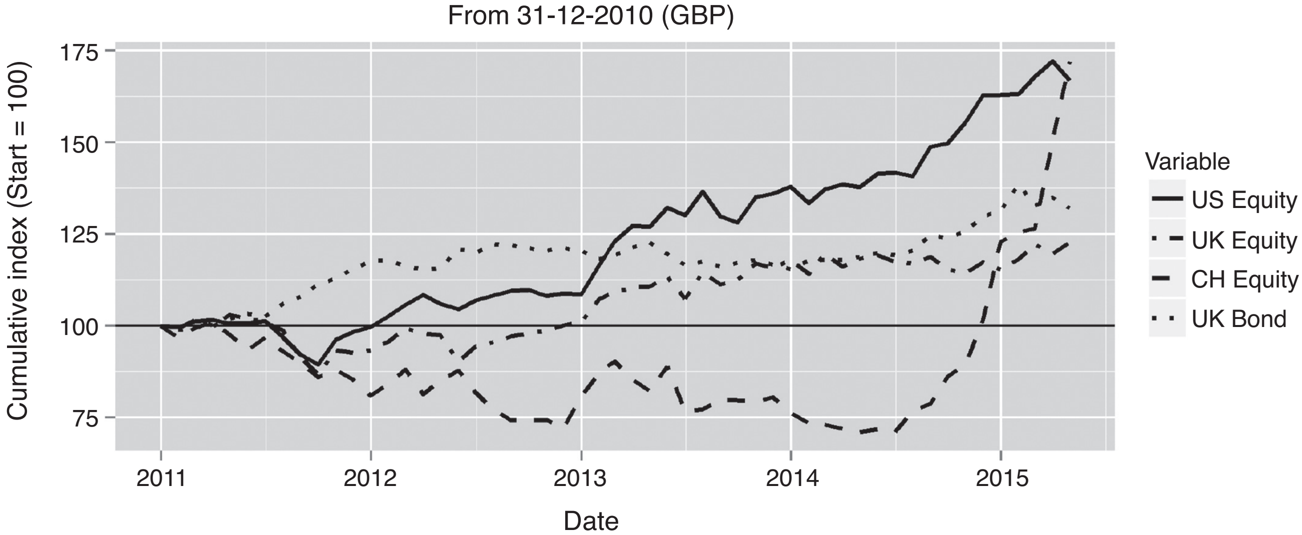

Figure 3: Cumulative index of the four assets from 31st December, 2010 until 30th April, 2015. The steep rise in the value of Chinese equities from June 2014 (as shown in Figure 2a) is readily apparent

semi-definite are constructed, and negative variances cannot arise.

Historical analysis seems a sensible approach to bounding likely ranges that correlations between assets may take. A precautionary approach might identify the largest correlations, and greatest changes compared with the current correlation matrix, so that a selected number of correlations (not all) are increased.

As the current analysis is based on 12-monthly returns, 12-monthly rolling correlations between asset pairs were used to determine the range of values each correlation could take, with thought being required as to how far back in the historical record analysis should go for the commencement of the rolling correlations.[28]

A problem might be the number of asset pairs that should have their rolling correlations explored. For the four asset portfolio, six correlations need to be explored.[29] An exhaustive analysis of a 20-asset portfolio would require computing 190 rolling correlations, which may prove prohibitive. A practitioner would likely have to form views as to which asset pairs were likely to be most significant and base adjustments to correlations on those selected.

In the current analysis, a limitation was the commencement of returns data for UK fixed interest,[30] which was 6th December, 2010; thus historical analysis of correlations could only commence from that date.[31] Index data to the final date of the initial sub-period (30th April, 2015[32]) is presented in Figure 3.

Rolling 12-month correlations between the returns on asset pairs are presented in Figure 4. We are mainly concerned with the maximum values the correlations could take, although the overall pattern may be of interest.

Table 2 shows maximum and minimum 12-month rolling correlation values for the six asset pairs, and the correlation over the initial sub-period. Examining Table 2, candidates for selection for use in stress-testing might be:

- Largest historical 12-month rolling correlations. The UK–US equity correlation at +0.89, noting that over the 12 months to 30th April, 2014, the UK–US correlation was at a historically low level of +0.16.
- The largest change (increase) in correlation 'Δ' between that used in the 12 months to 30th April, 2015 and the maximum 12-month rolling correlation. For UK–CH, $\Delta = 0.97$, an increase in correlation from −0.29 to +0.68.

Other criteria could be devised, and combinations tested to identify which have the largest impact on the results. For the current paper, it was sufficient to

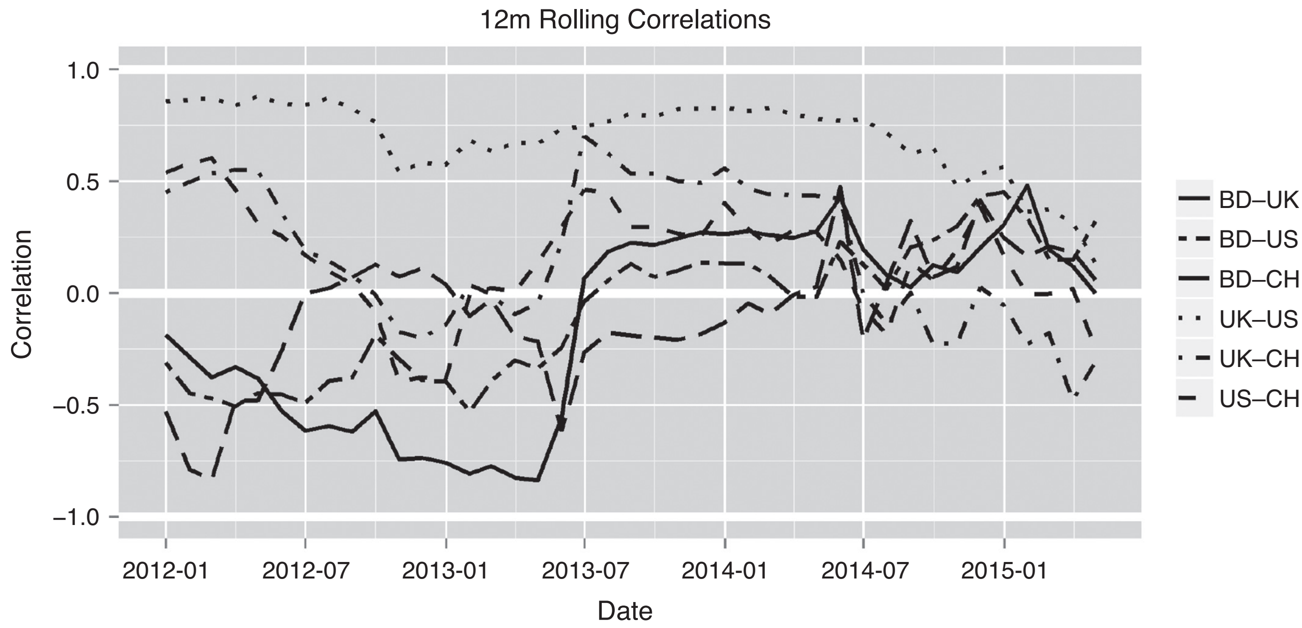

Figure 4: Twelve-month rolling correlations between asset pairs using returns data from 31st December, 2010 to 31st April, 2014. Dates are presented as at the end of the 12-month rolling window

Table 2: Minimum and maximum values of 12-month rolling correlations for the six asset pairs, 31st December, 2010–30th April, 2014

Asset pair	Minimum 12-month rolling correlation	Maximum 12-month rolling correlation (1)	12-month rolling correlation to 30th April, 2015 (2)	Δ, correlation (1) minus correlation (2)
BD–UK	–0.84	+0.48	–0.002	0.482
BD–US	–0.62	+0.43	+0.29	0.14
BD–CH	–0.84	+0.44	+0.06	0.38
UK–US	+0.16	**+0.89**	+0.16	0.73
UK–CH	–0.47	+0.68	–0.29	**0.97**
US–CH	–0.38	+0.60	–0.23	0.83
Maximum		**+0.89**		**0.97**

identify two correlations to change; both of the cases above used:

- UK–US equity correlation from +0.16 to +0.89
- UK–CH equity correlation from −0.29 to +0.68

Notice the potential unrealism of demanding high correlation between US and UK equities (+0.89), together with high correlation between UK and Chinese equities (+0.68), while leaving negative correlation between US and Chinese equities (−0.23). This point is returned to later.[33]

For the stress-test, the aim was to adjust the initial correlation matrix towards a target correlation matrix (targeted correlations to change are presented in bold face).

$$\widehat{\boldsymbol{R}}_{Target} = \begin{bmatrix} 1 & -0.0017 & 0.2941 & 0.0611 \\ -0.0017 & 1 & \mathbf{0.89} & \mathbf{0.68} \\ 0.2941 & \mathbf{0.89} & 1 & -0.2319 \\ 0.0611 & \mathbf{0.68} & -0.2319 & 1 \end{bmatrix}.$$

There is no guarantee that correlation matrix $\widehat{\boldsymbol{R}}_{Target}$ is positive semi-definite.[34] The purpose of this paper is to demonstrate two techniques to come up with a correlation matrix similar to (but not quite the same as) $\widehat{\boldsymbol{R}}_{Target}$ that will have the necessary mathematical properties.[35] The two techniques presented are those of Finger and N&B, which are covered in the following sections.

VARIANCE-COVARIANCE MATRIX STRESS-TEST FOLLOWING FINGER

Finger's method involves changing correlations by modifying selected return vectors.[36] Adjusting returns towards an average to increase correlation has intuitive appeal; however, a goal-seek algorithm is required. For a large multi-asset portfolio, if a long history of returns has to be modified this might become cumbersome. The example has returns on four assets (Table 1), covering 12 months.

Schachter's demonstration of this method[9] only adjusts a single correlation. Here, two correlations are increased. A single parameter, θ, is used to increase all selected correlations.[13] One cannot individually target two correlations to different values; instead increasing θ raises all selected correlations.[37] This is a weakness compared with N&B's method which permits differing targets for different correlations. In Finger's multiple correlation example, the average correlation was targeted. Here, the two targeted $\widehat{\boldsymbol{R}}_{Target}$ correlation values were 0.89 and 0.68, with average of 0.785. Selecting θ meant that the resulting adjusted correlation matrix had correlations between UK equity/US equity and UK equity/CH equity that averaged 0.785.

Appendix A details the method, with an outline and results following. As the correlation pairs to be adjusted were UK equity/US equity and UK equity/CH equity, returns from three assets were modified; UK equity, US equity and CH equity. Average returns across these three assets were used, generating modified returns (Table 3).

Modifying returns changes volatilities, so normalised returns can be used to restore original asset volatilities.[38] The normalised returns appear in Table 3, where asset volatilities now match the original data. The correlation matrix could be constructed from either modified, or normalised returns. As no adjustment has been required for the correlation between UK Bonds and another asset, the UK Bond returns were unchanged.

A search algorithm explored trial values of θ until the desired correlation target was obtained.[39] The average target correlation of 0.785 was sought for the average of the correlation pairs UK″/US″ and UK″/CH″.[40] When $\theta = 0.6408$, the correlations were $\rho_{\text{UK}''/\text{US}''} = 0.8198$ and $\rho_{\text{UK}''/\text{CH}''} = 0.7501$, with

Table 3: Results of Finger's method for modifying period returns

Date	UK Bond	UK Equity	US Equity	CH Equity	R_{ave}	UKE′	USE′	CHE′	UKE″	USE″	CHE″
31/05/2014	0.9017	0.9719	2.5667	0.9470	1.4952	1.3072	1.8801	1.2983	1.6326	2.4984	2.3754
30/06/2014	–0.5142	–1.4999	0.1485	–1.1093	–0.8202	–1.0643	–0.4722	–0.9240	–1.3293	–0.6275	–1.6907
31/07/2014	1.0592	–0.4047	–0.8872	8.0446	2.2509	1.2970	1.1237	4.3320	1.6199	1.4933	7.9261
31/08/2014	3.5049	1.5038	5.8208	3.1283	3.4843	2.7729	4.3236	3.3564	3.4632	5.7455	6.1411
30/09/2014	–0.6643	–2.9017	0.6385	8.7887	2.1752	0.3515	1.6232	4.5507	0.4391	2.1571	8.3263
31/10/2014	1.4272	–0.8622	3.8820	4.3254	2.4484	1.2592	2.9634	3.1226	1.5727	3.9380	5.7133
30/11/2014	3.2403	2.5649	4.7312	13.2434	6.8465	5.3086	6.0867	9.1443	6.6300	8.0885	16.7308
31/12/2014	1.3474	–1.6859	0.3135	21.2640	6.6305	3.6433	4.3614	11.8868	4.5502	5.7958	21.7487
31/01/2015	5.3792	2.5213	–0.0789	2.0892	1.5105	1.8736	0.9396	1.7184	2.3400	1.2486	3.1441
28/02/2015	–4.2004	3.3809	3.0105	0.7217	2.3711	2.7338	2.6007	1.7786	3.4143	3.4561	3.2542
31/03/2015	1.5985	–2.1548	2.1708	18.3803	6.1321	3.1555	4.7092	10.5317	3.9410	6.2580	19.2693
30/04/2015	–2.0978	2.6335	–2.7236	14.7430	4.8843	4.0758	2.1515	8.4255	5.0904	2.8592	15.4158
Sample SD	2.5719	2.1782	2.4945	7.4770		1.7441	1.8771	4.0866	2.1782	2.4945	7.4770
Vol %pa	8.9094	7.5456	8.6413	25.9013		6.0416	6.5026	14.1564	7.5456	8.6413	25.9013
					Vol Difference	–1.5040	–2.1386	–11.7448	0.0000	0.0000	0.0000

Notes: UKE′ denotes R'_{UKE}, UKE″ denotes R''_{UKE} (see main text), with similar interpretations for USE and CHE.

average 0.785 as required. The individual correlation targets of 0.89 and 0.68 have not been met, which is a limitation of the method. The resulting adjusted correlation matrix was[41]:

$$\widehat{\boldsymbol{R}}_{\boldsymbol{Finger}} = \begin{bmatrix} 1 & 0.1446 & 0.2754 & 0.1022 \\ 0.1446 & 1 & \mathbf{0.8198} & \mathbf{0.7501} \\ 0.2754 & \mathbf{0.8198} & 1 & 0.7229 \\ 0.1022 & \mathbf{0.7501} & 0.7229 & 1 \end{bmatrix}.$$

As the normalised returns of three of the asset classes differ from the original returns, correlations between UK Bonds and other asset classes were also changed, as well as the correlation between US equity and CH equity, which has risen from −0.23 to +0.72.[42]

The adjusted correlation matrix $\widehat{\boldsymbol{R}}_{\boldsymbol{Finger}}$ was used to calculate a stress-test parametric VaR. The portfolio asset weights were left unchanged,[43] and normalised returns ensure asset volatilities were unchanged. Thus:

$$\widehat{\boldsymbol{S}}_{\boldsymbol{Finger}} = \mathbf{v}\widehat{\boldsymbol{R}}_{\boldsymbol{Finger}}\mathbf{v}$$

$$\widehat{\sigma}^2_{Finger} = \vec{w}^T \widehat{\boldsymbol{S}}_{\boldsymbol{Finger}} \vec{w}$$

$$\widehat{\sigma}_{Finger} = 7.80\% pa$$

$$VaR_{\%} = \bullet - N\widehat{\sigma}_{Finger}\, \delta t^{1/2} \bullet = 3.70\%$$

The 95 per cent monthly parametric VaR for the stress-test would be 3.70 per cent. Applying a multiplier of four to the portfolio volatility, $\widehat{\sigma}'_{Finger} = 4 \times 7.80 = 31.19\% pa$, with resulting 95 per cent monthly stress-test parametric VaR of 14.81 per cent, significantly higher than the unstressed VaR of 2.42 per cent.

VARIANCE-COVARIANCE MATRIX STRESS-TEST FOLLOWING N&B

Numpacharoen and Bunwong (N&B) propose a different approach to adjusting a correlation matrix. The correlation matrix is adjusted directly with Cholesky decomposition, ensuring the resulting matrix is positive semi-definite. Products of trigonometrical functions represent correlations, using changes to correlative angles, ensuring correlations lie within $-1 \leq \rho \leq +1$. This has advantages. First, only the correlation matrix is required, which is helpful compared with Finger's approach for larger portfolios if many asset return histories have to be adjusted, or if historical data are unavailable. Secondly, Finger's approach only permits multiple correlations to be adjusted in a blanket manner, with a single parameter θ changing all correlations targeted. Thus multiple correlations cannot be targeted to unique values. N&B's approach permits individual correlations to be individually targeted. The disadvantage of N&B's method is the additional mathematical complexity, which is a reason for presenting a fully worked example in the current paper. Details of the computational procedure are also extensively presented elsewhere[44] (see also Appendix B and below).

In the original correlation matrix, assets were listed in the order UK Bonds, UK equity, US equity and Chinese equity. The target correlation matrix was[45]

$$\widehat{\boldsymbol{R}}_{\boldsymbol{Target}} = \begin{bmatrix} 1 & -0.0017 & 0.2941 & 0.0611 \\ -0.0017 & 1 & \mathbf{0.89} & \mathbf{0.68} \\ 0.2941 & \mathbf{0.89} & 1 & -0.2319 \\ 0.0611 & \mathbf{0.68} & -0.2319 & 1 \end{bmatrix}.$$

The assets are divided into two groups. Group A contains the assets with correlations to be adjusted, while group B contains assets with correlations not to be adjusted.

- Group A: UK, US, CH
- Group B: UKB

The initial correlation matrix should be arranged into submatrices as follows:

$$\boldsymbol{C}_{Initial} = \begin{bmatrix} \boldsymbol{C}_{AA} & \boldsymbol{C}_{AB} \\ \boldsymbol{C}_{BA} & \boldsymbol{C}_{BB} \end{bmatrix}.$$

$\boldsymbol{C}_{AA}$ and $\boldsymbol{C}_{BB}$ are correlation matrices among the assets in groups A and B, respectively, while $\boldsymbol{C}_{AB}$ and $\boldsymbol{C}_{BA}$ are correlations between assets in group A and assets in group B. In the current example, there are three assets in group A and one asset in group B; thus $\boldsymbol{C}_{AA}$ is a 3×3 matrix and $\boldsymbol{C}_{BB}$ is 1×1. The order of assets in the initial correlation matrix needs to be changed as shown in Table 4.[46]

$$\boldsymbol{C}_{\boldsymbol{Initial}} = \begin{bmatrix} 1 & 0.1553 & -0.2914 & -0.0017 \\ 0.1553 & 1 & -0.2319 & 0.2941 \\ -0.2914 & -0.2319 & 1 & 0.0611 \\ -0.0017 & 0.2941 & 0.0611 & 1 \end{bmatrix}.$$

Table 4a: Initial correlation matrix with asset classes shown

Initial	UKB	UKE	USE	CHE
UKB	1	–0.0017	0.2941	0.0611
UKE	–0.0017	1	0.1553	–0.2914
USE	0.2941	0.1553	1	–0.2319
CHE	0.0611	–0.2914	–0.2319	1

Table 4b: Initial correlation matrix after reordering of asset classes

Reordered	UKE	USE	CHE	UKB
UKE	1	0.1553	–0.2914	–0.0017
USE	0.1553	1	–0.2319	0.2941
CHE	–0.2914	–0.2319	1	0.0611
UKB	–0.0017	0.2941	0.0611	1

One cannot generally expect the ordering of the assets to be convenient, so the process is illustrated here. Reordering the target correlation matrix,[47]

$$\widehat{\boldsymbol{C}}_{Target} = \begin{bmatrix} 1 & \mathbf{0.89} & \mathbf{0.68} & -0.0017 \\ \mathbf{0.89} & 1 & -0.2319 & 0.2941 \\ \mathbf{0.68} & -0.2319 & 1 & 0.0611 \\ -0.0017 & 0.2941 & 0.0611 & 1 \end{bmatrix}.$$

Cholesky decomposition is used on $\boldsymbol{C}_{Initial}$ (see Appendix B), so that the Hermitian positive definite matrix may be decomposed into a lower triangular matrix and its transpose $\boldsymbol{C}_{Initial} = \boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}$. Here[48]

$$\boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ +0.1553 & +0.9879 & 0 & 0 \\ -0.2914 & -0.1890 & +0.9378 & 0 \\ -0.0017 & +0.2979 & +0.1247 & +0.9464 \end{bmatrix}.$$

With a resulting correlative angles matrix (radians),[49]

$$\theta_{4\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \theta_{21} & 0 & 0 & 0 \\ \theta_{31} & \theta_{32} & 0 & 0 \\ \theta_{41} & \theta_{42} & \theta_{43} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1.4149 & 0 & 0 & 0 \\ 1.8665 & 1.7697 & 0 & 0 \\ 1.5725 & 1.2683 & 1.4398 & 0 \end{bmatrix}.$$

These angles will later be used to reconstruct the amended correlation matrix.

Cholesky decomposition is repeated on submatrix $\boldsymbol{A}$ of target correlations.[50]

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0.89 & 0.68 \\ 0.89 & 1 & -0.2319 \\ 0.68 & -0.2319 & 1 \end{bmatrix}.$$

The eigenvalues of $\boldsymbol{A}$ are −0.2389, 1.2230 and 2.0159, making $\boldsymbol{A}$ not positive semi-definite and not a valid correlation matrix. (The results of naively proceeding with the Cholesky decomposition are outlined in Appendix C.) N&B give two worked examples: the first uses only a single target correlation; in their second example they target multiple correlations, but all to the same value (0.85), while the current paper targets two different correlation values (0.89 and 0.68). N&B state that the target submatrix ($\boldsymbol{C}_{AA}$, or $\boldsymbol{A}$ as shown above) must itself be a valid correlation matrix,[51] without suggesting how to achieve that. The current example with two different target correlations reveals a possible difficulty with their method. As the method suggested by Finger could be used to target multiple correlations towards the same value, it reduces the usefulness of N&B's approach if multiple correlations cannot be targeted to different values.

Here it is helpful to introduce an element of practitioner 'common sense': what is actually being asked for? The original correlations (two decimal places) were:

- UK equity–US equity: 0.16
- UK equity–Chinese equity: −0.29
- Chinese equity–US equity: −0.23

A decision was made to arbitrarily increase the correlations of two of these to

- UK equity–US equity: 0.89
- UK equity–Chinese equity: 0.68

While retaining the original correlation of −0.23 between Chinese and US equity. With positive correlations between US and UK equities, and UK and Chinese equities, it may seem unreasonable to demand a negative correlation between US and Chinese equities given the common link of UK equities.

Something is required to determine an acceptable[52] correlation between US and Chinese equities for the target correlations that were selected, given the UK equities link. This is explored in Appendix D; however, at this stage the author proposes a convenient rule for linked correlations. The subscripts denote different assets and an acceptable correlation between assets 1 and 3 is to be deduced given correlations between assets 1 and 2, and 2 and 3.[53]

$$\rho_{23} = \rho_{12}\rho_{13} \pm \sqrt{1-\rho_{12}^2} \times \sqrt{1-\rho_{13}^2}.$$

Here,

$$\rho_{USCH} = \rho_{USUK}\rho_{UKCH} \pm \sqrt{1-\rho_{USUK}^2} \times \sqrt{1-\rho_{UKCH}^2},$$

$$\begin{aligned}\rho_{USCH} &= 0.89 \times 0.68 \pm \sqrt{1-0.89^2} \\ &\quad \times \sqrt{1-0.68^2} \\ &= 0.6052 \pm \sqrt{0.1118} \\ &= 0.2709 \text{ or } 0.9395.\end{aligned}$$

As the correlation between US and Chinese equities was not directly targeted, it appears sensible to adopt the value of ρ_{USCH} nearest to the original value −0.23, ie $\rho_{USCH} = 0.2709$. Thus target submatrix $\boldsymbol{A}$ becomes:

$$\boldsymbol{A'} = \begin{bmatrix} 1 & 0.89 & 0.68 \\ 0.89 & 1 & 0.2709 \\ 0.68 & 0.2709 & 1 \end{bmatrix}.$$

The eigenvalues of $\boldsymbol{A'}$ are 0, 0.7398 and 2.260, so it is positive semi-definite and a valid correlation matrix, the negative eigenvalue being replaced by a zero. Using Cholesky decomposition, re-express

$\boldsymbol{A'} = \boldsymbol{VV^T}$, where

$$\boldsymbol{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0.89 & 0.4560 & 0 \\ 0.68 & -0.7332 & 0 \end{bmatrix}.$$

The lower triangular matrix was used to obtain the correlative angles, $\boldsymbol{V} = \boldsymbol{B_{3\times3}}$.

$$\boldsymbol{V} = \begin{bmatrix} 1 & 0 & 0 \\ \cos\theta_{21} & \sin\theta_{21} & 0 \\ \cos\theta_{31} & \cos\theta_{32}\sin\theta_{31} & \sin\theta_{31}\sin\theta_{32} \end{bmatrix}.$$

Resulting in a matrix of correlative angles,[54]

$$\theta_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ \theta_{21} & 0 & 0 \\ \theta_{31} & \theta_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0.4735 & 0 & 0 \\ 0.8230 & 3.1416 & 0 \end{bmatrix}.$$

Angle matrix $\theta_{3\times3}$ contains information about the target correlations to be adjusted, and originated from the submatrix at the top left of the initial correlation matrix. Substituting $\theta_{3\times3}$ as a submatrix into the top left of $\theta_{4\times4}$ gives revised correlative angle matrix $\widehat{\theta}_{4\times4}$.

$$\widehat{\theta}_{4\times4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{0.4735} & 0 & 0 & 0 \\ \mathbf{0.8230} & \mathbf{3.1416} & 0 & 0 \\ 1.5725 & 1.2683 & 1.4398 & 0 \end{bmatrix}.$$

The non-zero elements substituted from $\theta_{3\times3}$ are shown in bold face. The angles of $\widehat{\theta}_{4\times4}$ used in $\boldsymbol{B_{4\times4}}$ generate a new lower diagonal matrix= $\widehat{\boldsymbol{U}}$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \cos\widehat{\theta}_{21} & \sin\widehat{\theta}_{21} & 0 & 0 \\ \cos\widehat{\theta}_{31} & \cos\widehat{\theta}_{32}\sin\widehat{\theta}_{31} & \sin\widehat{\theta}_{31}\sin\widehat{\theta}_{32} & 0 \\ \cos\widehat{\theta}_{41} & \cos\widehat{\theta}_{42}\sin\widehat{\theta}_{41} & \cos\widehat{\theta}_{43}\sin\widehat{\theta}_{41}\sin\widehat{\theta}_{42} & \sin\widehat{\theta}_{41}\sin\widehat{\theta}_{42}\sin\widehat{\theta}_{43} \end{bmatrix} = \widehat{\boldsymbol{U}}.$$

$$\widehat{\boldsymbol{U}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.89 & 0.4560 & 0 & 0 \\ 0.68 & -0.7332 & 0 & 0 \\ -0.0017 & 0.2979 & 0.1247 & 0.9464 \end{bmatrix}.$$

Finally, $\widehat{\boldsymbol{U}}\widehat{\boldsymbol{U}}^{\boldsymbol{T}}$ recreates the adjusted correlation matrix.

$$\widehat{\boldsymbol{U}}\widehat{\boldsymbol{U}}^{\boldsymbol{T}} = \widehat{\boldsymbol{R}}'_{N\&B} = \begin{bmatrix} 1 & 0.89 & 0.68 & -0.0017 \\ 0.89 & 1 & 0.2709 & 0.1344 \\ 0.68 & 0.2709 & 1 & -0.2196 \\ -0.0017 & 0.1344 & -0.2196 & 1 \end{bmatrix}.$$

This has eigenvalues 0, 0.5828, 1.1560 and 2.2610, making it positive semi-definite and a valid correlation matrix. For comparison with the original

Table 5a: Adjusted correlation matrix with reordered asset classes

Reordered	UKE	USE	CHE	UKB
UKE	1	0.89	0.68	–0.0017
USE	0.89	1	0.2709	0.1344
CHE	0.68	0.2709	1	–0.2196
UKB	–0.0017	0.1344	–0.2196	1

Table 5b: Adjusted correlation matrix with asset classes in initial order

Initial order	UKB	UKE	USE	CHE
UKB	1	–0.0017	0.1344	–0.2196
UKE	–0.0017	1	**0.89**	**0.68**
USE	0.1344	**0.89**	1	0.2709
CHE	–0.2196	**0.68**	0.2709	1

correlation matrix, the order of the assets needs to be rearranged, reversing the process of Table 4; see Table 5. Thus,

$$\widehat{\boldsymbol{R}}_{\boldsymbol{N\&B}} = \begin{bmatrix} 1 & -0.0017 & 0.1344 & -0.2196 \\ -0.0017 & 1 & \mathbf{0.89} & \mathbf{0.68} \\ 0.1344 & \mathbf{0.89} & 1 & 0.2709 \\ -0.2196 & \mathbf{0.68} & 0.2709 & 1 \end{bmatrix}.$$

Comparing this with initial correlation matrix, $\boldsymbol{R}_{\boldsymbol{Initial}}$, and target correlation matrix, $\widehat{\boldsymbol{R}}_{\boldsymbol{Target}}$, it can be seen that N&B's method has generated both targeted correlations exactly, which Finger's approach did not.

The stressed correlation matrix $\widehat{\boldsymbol{R}}_{\boldsymbol{N\&B}}$ is used to calculate a stressed parametric VaR.[55] Thus,

$$\widehat{\boldsymbol{S}}_{\boldsymbol{N\&B}} = \mathbf{v}\widehat{\boldsymbol{R}}_{\boldsymbol{N\&B}}\mathbf{v}$$

$$\widehat{\sigma}^2_{N\&B} = \vec{w}^T \widehat{\boldsymbol{S}}_{\boldsymbol{N\&B}} \vec{w}$$

$$\widehat{\sigma}_{N\&B} = 6.82\%\, pa$$

$$VaR_{\%} = \bullet -N\widehat{\sigma}_{N\&B}\,\delta t^{1/2} \bullet = 3.24\%$$

The 95 per cent monthly parametric VaR for the stress-test might be 3.24 per cent. Applying a multiplier of four to portfolio volatility, $\widehat{\sigma}'_{N\&B} = 4 \times 6.82 = 27.29\%\, pa$, with a resulting 95 per cent monthly stress-test parametric VaR of 12.96 per cent, is significantly higher than the unstressed VaR of 2.42 per cent.[56]

COMPARISON BETWEEN FINGER AND N&B

Previous sections have explored the generation of a stressed volatility and parametric VaR for a four-asset portfolio using hypothetical variance-covariance matrix stress-test methods proposed respectively by Finger and N&B. Results are summarised in Table 6.

The stressed portfolio volatility for Finger's method (7.80 per cent p.a.) is higher than that obtained by N&B's method, 6.82 per cent. This results in higher stressed values using Finger's method for the 95 per cent monthly parametric VaRs, as well as when based on four times the volatility. Interpretation appears straightforward; the average of the elements of Finger's adjusted correlation matrix is +0.60, but +0.47 under N&B's method. Finger's approach has generated higher average correlations overall, resulting in a greater loss of diversification.

Higher risk may not indicate a better result, however. Excessive assumptions, making a portfolio appear unduly risky, may result in overly cautious management, with consequences for returns. Alternatively consider how closely the adjusted correlation matrices match the target correlation matrix, which was devised by the practitioner following analysis of correlation data and presumably reflects their primary concerns. Unnecessary departures from this would likely be deemed unhelpful.

Comparing the averages of the elements of the correlation matrices, the average correlation for the initial matrix was +0.25, while the average for the target correlation matrix was +0.46, which presumably encapsulates the practitioner's primary concerns. Finger's adjusted matrix had an average correlation of +0.60, while N&B's adjusted matrix average was +0.47, essentially the same as the target matrix average.

Other comparisons of the adjusted correlation matrices with the target indicate similar results. Comparing the magnitudes of the differences between the elements in adjusted matrices and the target matrix, the average difference is 0.16 for Finger, and a lesser 0.12 for N&B. The maximum

Table 6: Comparison between the variance-covariance stress-test calculations of Finger and N&B

	Finger	N&B
Initial correlation matrix (average correlation 0.25)	$\begin{bmatrix} 1 & -0.0017 & 0.2941 & 0.0611 \\ -0.0017 & 1 & 0.1553 & -0.2914 \\ 0.2941 & 0.1553 & 1 & -0.2319 \\ 0.0611 & -0.2914 & -0.2319 & 1 \end{bmatrix}$	
Portfolio volatility		5.088%p.a.
95% monthly parametric VaR		2.42%
Target correlation matrix (average correlation 0.46)	$\begin{bmatrix} 1 & -0.0017 & 0.2941 & 0.0611 \\ -0.0017 & 1 & \mathbf{0.89} & \mathbf{0.68} \\ 0.2941 & \mathbf{0.89} & 1 & -0.2319 \\ 0.0611 & \mathbf{0.68} & -0.2319 & 1 \end{bmatrix}$	
Adjusted correlation matrix	$\begin{bmatrix} 1 & 0.1446 & 0.2754 & 0.1022 \\ 0.1446 & 1 & \mathbf{0.8198} & \mathbf{0.7501} \\ 0.2754 & \mathbf{0.8198} & 1 & 0.7229 \\ 0.1022 & \mathbf{0.7501} & 0.7229 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -0.0017 & 0.1344 & -0.2196 \\ -0.0017 & 1 & \mathbf{0.89} & \mathbf{0.68} \\ 0.1344 & \mathbf{0.89} & 1 & 0.2709 \\ -0.2196 & \mathbf{0.68} & 0.2709 & 1 \end{bmatrix}$
Average correlation	0.60	0.47
Difference between target and adjusted correlation matrices	$\begin{bmatrix} 0 & 0.15 & -0.02 & 0.04 \\ 0.15 & 0 & \mathbf{-0.07} & \mathbf{0.07} \\ -0.02 & \mathbf{-0.07} & 0 & 0.95 \\ 0.04 & \mathbf{0.07} & 0.95 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & -0.16 & -0.28 \\ 0 & 0 & \mathbf{0} & \mathbf{0} \\ -0.16 & \mathbf{0} & 0 & 0.50 \\ -0.28 & \mathbf{0} & 0.50 & 0 \end{bmatrix}$
Average magnitude of differences	0.16	0.12
Maximum magnitude of differences	0.95	0.50
Stressed volatility	7.80%p.a.	6.82%p.a.
Stressed 95% monthly parametric VaR	3.70%	3.24%
4x stressed volatility	31.19%p.a.	27.29%p.a.
95% monthly parametric VaR based on 4x stressed volatility	14.81%	12.96%

values of these differences demonstrate a clear outcome: for Finger's approach the maximum difference from the target was 0.95, whereas for N&B it was only 0.5. These relate to the problematic correlation between US equity and Chinese equity, set at −0.23 in the target. For this pair Finger's method generated an adjusted correlation of +0.72, while N&B's method, with the additional input described above,[57] only produced +0.27. In this case, N&B's method with the small addition above would appear to have matched the target correlation matrix better than Finger's method.

While stressed volatilities were not used in the current example for simplicity, both methods allow volatility adjustments. In Finger's method, instead of normalising modified returns to restore original asset volatilities, modified returns could be 'normalised' to other values as desired. With N&B's method, adjustment to volatilities is simple, as the original volatility matrix was used in conjunction with the adjusted correlation matrix to obtain the adjusted variance-covariance matrix in the parametric VaR calculation. The practitioner could simply replace it with a matrix of changed volatilities to explore their impact on stress-test results.

VAR ESTIMATE FOR PERIOD ENCOMPASSING CORRECTION

Finally the portfolio parametric VaR was calculated over the terminal sub-period from 31st August, 2014 to 31st August, 2015, the period including the Chinese equities correction. Twelve-monthly returns were calculated for each asset (Table 7). While this comparison is included for completeness, the reader should be aware that, as very small datasets have been used (12-monthly returns), the results should not be regarded as being authoritative as a comparison between a portfolio stress-test and an ensuing market correction. The objective (as with the rest of this paper) was to illustrate and explore the method within the framework of a worked example that is capable of being replicated by the interested reader.

Calculation of the parametric VaR proceeded as before. The correlation matrix was (with assets presented in the order listed in Table 7):

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0.1278 & 0.2848 & 0.1272 \\ 0.1278 & 1 & 0.6060 & 0.2262 \\ 0.2848 & 0.6060 & 1 & 0.3442 \\ 0.1272 & 0.2262 & 0.3442 & 1 \end{bmatrix}.$$

Correlations for this final sub-period were all positive; however, none reached the +0.89 correlation proposed in the stress-test. Neglecting the unit correlations down the leading diagonal, the highest correlation (+0.61) was between UK and US equities; other correlations were all much lower, none being greater than +0.34 (US and China equities). In this sense the stress test scenario was tougher than actually occurred.

The variance-covariance matrix was calculated and the same asset weights used. The resulting portfolio variance gave a portfolio volatility for the terminal sub-period of 9.30 per cent p.a., with 95 per cent monthly parametric VaR of 4.42 per cent.[58]

The volatility observed during the terminal period (9.30 per cent) was higher than the stressed volatilities derived from the adjusted correlation matrices (6.82 and 7.80 per cent p.a.), meaning that the 95 per cent monthly parametric VaRs were also less than the 4.42 per cent above, at 3.24 per cent and 3.70 per cent, respectively. The actual volatility was a factor 1.2–1.4 times larger than the stressed volatilities above (with the same factor feeding through to the parametric VaRs), well within the stressed volatility and VaR estimates based on a multiplier of four times the stressed volatility (with stress-test volatilities of 27.29 per cent p.a. and 31.19 per cent p.a., together with stressed parametric VaR estimates of 12.96 per cent and 14.81 per cent, notably higher than those observed in the terminal period).

One would not expect a stress-test to estimate the risk measures during a stressed period exactly, but to generate an envelope that future risk values would be expected not to exceed. This shows the importance of using a multiplier on the stressed values derived (here a factor of four). Clearly the volatility and parametric VaR lie well within this envelope as one would hope, as the market correction period should not necessarily be assumed to be the 'worst' event that could reasonably occur.

Table 7: Percentage monthly returns for the terminal sub-period, 31st August, 2014–31st August, 2015

Date	UK Bond	UK Equity	US Equity	CH Equity
30/09/2014	–0.6643	–2.9017	0.6385	8.7887
31/10/2014	1.4272	–0.8622	3.8820	4.3254
30/11/2014	3.2403	2.5649	4.7312	13.2434
31/12/2014	1.3474	–1.6859	0.3135	21.2640
31/01/2015	5.3792	2.5213	–0.0789	2.0892
28/02/2015	–4.2004	3.3809	3.0105	0.7217
31/03/2015	1.5985	–2.1548	2.1708	18.3803
30/04/2015	–2.0978	2.6335	–2.7236	14.7430
31/05/2015	0.3367	0.2843	1.6242	4.3164
30/06/2015	–1.7466	–5.3085	–4.7511	–9.7650
31/07/2015	1.5991	2.3024	2.7275	–15.1652
31/08/2015	0.2445	–5.9716	–4.9709	–13.7657

Table 8: Initial and terminal sub-period volatilities (%p.a.)

Volatility %p.a.	UK bond	UK equity	US equity	CH equity
Initial sub-period to 30/04/2015	8.91%	7.55%	8.64%	25.90%
Terminal sub-period to 31/08/2015	8.78%	11.20%	11.10%	41.98%

Equally, the exact volatility and VaR for the correction period would not be estimated, as one generally cannot hope to correctly estimate the exact correlations and volatilities that will occur. In this case the stress test has overestimated some correlations and underestimated others.

A further reason that the stress-test would not be expected to replicate the terminal period volatility and VaR is that no changes were made to the asset volatilities in the stress-test, an area that could also have been addressed, perhaps using similar approaches to those used to explore likely correlation ranges applied to volatility (Table 8).[59]

The terminal period had higher equity volatilities than those included in the stress-tests. This would be an obvious way of extending the stress-test scenario (the volatility of UK bonds remained essentially unchanged).

SUMMARY AND DISCUSSION

Both the hypothetical variance-covariance matrix stress-test examples presented generated a risk envelope wider than the risk estimates (volatility and parametric VaR) observed during a following market reversal. This seems reasonable; one might not expect a market reversal to represent the 'extreme but plausible' scenario stress-testing is intended to capture. Both stress-tests required the use of a multiplying factor to volatility[60]; otherwise they would have indicated a risk magnitude less than occurred during the reversal.[61]

Table 9 presents a summary of key aspects of the two methods. Finger's approach has intuitive appeal with returns adjusted towards an average to increase correlation, but a goal-seek algorithm is required. For a large multi-asset portfolio, with a long history of returns,[62] this might become cumbersome and a complete asset return history may not be available. In this case N&B's approach seems practical, as only the correlation matrix is required, although the mathematical sophistication may discourage some practitioners. Although N&B's method ensures an acceptable correlation matrix, there is no guarantee of economic validity. Choice between the methods may be dictated by availability of asset returns for Finger, and access to a Cholesky decomposition algorithm[63] for N&B.

Adjusting correlations between assets revealed differences between the approaches. Finger uses a single parameter, θ, to increase the values of all selected correlations. Increases to multiple correlations only targeted some average, rather than different correlations for different asset pairings. This is weak if a practitioner has different correlation targets in mind for different asset pairs. N&B's method allowed different asset pairs to be adjusted to different correlation values, appearing superior. In the example above, however, with correlations linked by a common asset,[64] a non-positive definite matrix was targeted. A work-around was proposed, giving a mathematically (and intuitively) acceptable correlation.

Both methods worked straightforwardly in conjunction with estimates of stressed parametric VaRs. They permit practitioners to generate correlation matrices with suitable mathematical properties, without depending on correlations generated from historical events. This benefits flexibility in scenario creation. Regarding isolating specific concerns, historical events tend to be 'messy' with many knock-on effects, while these methods permit a focus on individual portfolio aspects. Similarly, to explore extreme events, historical methods only permit this if suitable events lie within the historical record, while the hypothetical methods permit factors to be pushed further.

By adjusting correlation matrices, such approaches might lend themselves to studies linked to portfolio optimisations requiring outcomes in terms of both risks and returns; this aspect has not been explored.

Data availability is a practical consideration. Finger's approach would likely require considerable asset returns data, which may not be conveniently

Table 9: Key aspects of hypothetical variance-covariance matrix stress-testing using the approaches of Finger or N&B

Aspect	Finger	N&B
Does the approach generate acceptable correlation matrices? (Matrices are positive semi-definite; non-targeted correlation values are adjusted to ensure this.)	Yes	Yes
Can historical correlation matrices be used as a guide?	Yes	Yes
Can both correlations and volatilities be adjusted?	Yes	Yes
Can correlations be individually adjusted to values selected by the practitioner, to isolate specific concerns?	Yes	Yes
Can multiple correlations be adjusted?	Yes	Yes
With multiple correlations, can individual correlations be adjusted to different values?	No	Yes
Does a single control parameter raise (lower) all targeted correlations?	Yes	No
Are period returns vectors required for all portfolio assets, with calculations required on these returns?	Yes	No
Can the correlation matrix be adjusted directly?	No	Yes
Is a trial-and-error search algorithm required?	Yes	No
Is a Cholesky decomposition algorithm required?	No	Yes
What is the level of mathematical complexity required?	Lower	Higher
What level of data availability is required?	Higher	Lower
Did stress-test generate risk estimates that were greater than observed during the following market correction presented without using an additional factor on volatilities?	No	No
Did stress-test generate risk estimates that were greater than observed during the following market correction presented when an additional factor on volatilities was used? (In the current study a factor of 4 was used.)	Yes	Yes
Should a multiplying factor be applied to resulting portfolio volatilities for the stress-test? (In the current study a factor of 4 was used.)	Yes	Yes

available. Extensive explorations of the likely ranges of correlations may also be limited by restricted data. While limited data for determining correlation ranges would impact scenario selection for both approaches, N&B's method may ease practitioner exploration of correlation concerns, even based on limited evidence. This is because N&B's method only requires the current portfolio correlation matrix coupled with the practitioner's views.

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References and notes

1 For a working definition of stress-testing and a classification of methodologies, see Rayer Q. G. (2015) 'Dissecting portfolio stress-testing', *Review of Financial Markets*, Vol. 7, pp. 2–7.

2 Kupiec, P. (1998) 'Stress testing in a value at risk framework', *Journal of Derivatives*, Vol. 6, pp. 7–24.

3 For definitions of types of risk, see Choudhry, M. (2000) 'An introduction to value at risk', Securities Institute Services Ltd, London; and Crouhy, M., Galai, D. and Mark, R. (2014) 'The essentials of risk management', 2nd edn, McGraw-Hill Education, New York.

4 Various terms are used in the literature rather loosely. See Crouhy, M., Galai, D. and Mark, R. (2014) 'The essentials of risk management', 2nd edn, McGraw-Hill Education, New York; Rayer, Q. G. (2015) 'Dissecting portfolio stress-testing', *Review of Financial Markets*, Vol. 7, pp. 2–7; Schachter, B. (2004) 'Stress testing', in C. Alexander and E. Sheedy (eds), The professional risk managers' handbook, PRMIA Publications, Wilmington, DE.

5 The classification of Rayer, Q. G. (2015) 'Dissecting portfolio stress-testing', *Review of*

Financial Markets, Vol. 7, pp. 2–7 further divides artificial stress-tests into 'hypothetical' stress-tests and 'algorithmic' stress-tests. Algorithmic stress-tests do not concern us here, as this paper explores only hypothetical stress-testing. For further information on algorithmic stress-tests see Schachter, B. (2004) 'Stress testing', in C. Alexander and E. Sheedy (eds), The professional risk managers' handbook, PRMIA Publications, Wilmington, DE.

6 Crouhy, M., Galai, D. and Mark, R. (2014) 'The essentials of risk management', 2nd edn, McGraw-Hill Education, New York.

7 Actually a number of academic studies debate this point; a discussion can be found in Schachter, B. (2004) 'Stress testing', in C. Alexander and E. Sheedy (eds), The professional risk managers' handbook, PRMIA Publications, Wilmington, DE.

8 The volatility matrix is constructed by placing the volatilities of the assets down the leading diagonal.

9 Schachter, B. (2004) 'Stress testing', in C. Alexander and E. Sheedy (eds), The professional risk managers' handbook, PRMIA Publications, Wilmington, DE.

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11 Higham, N. J. (2002) 'Computing the nearest correlation matrix — a problem from finance', *IMA Journal of Numerical Analysis*, Vol. 22, pp. 329–343. The Frobenius norm of a matrix is the square root of the sum of the absolute squares of the matrix elements. If a correlation matrix is subtracted from some target correlation matrix this gives a matrix of 'errors' and the Frobenius norm a measure of their magnitude.

12 Turkay, S., Epperleinm E. and Christofides, N. (2003) 'Correlation stress testing for value-at-risk', *Journal of Risk*, Vol. 5, No. 4, pp. 75–89.

13 Finger, C. A. (1997) 'Methodology to stress correlations', *RiskMetrics Monitor*, 4th Quarter, pp. 3–11.

14 Numpacharoen, K. and Bunwong, K. (2012) 'An intuitively valid algorithm for adjusting the correlation matrix in risk management and option pricing', available at: SSRN-id1980761.

15 This ensures correlations lie within $-1 \leq \rho_{ij} \leq +1$ and the resulting adjusted correlation matrix has the necessary mathematical properties.

16 Exchange traded fund (ETF).

17 The tickers required for the assets were: 'GILS.PA', '^FTAS', '^GSPC' and '000001.SS', *Yahoo Finance*, available at: http://finance.yahoo.com/ (accessed October 2015).

18 Exchange rate pairs US$-GBP and CNY-GBP were required, conveniently represented in Oanda by tickers 'USD/GBP' and 'CNY/GBP'. Oanda (n.d.) 'Historical exchange rates', *oanda.com*, available at: http://www.oanda.com/currency/historical-rates/ (accessed October 2015).

19 It should be noted that these databases can be subject to periodic revisions (as indeed occurred during the preparation of this paper). Thus, should the reader wish to reproduce the calculations herein in detail, they would be advised to check that a more recent data extraction does not differ from the returns data presented.

20 Weekly or daily returns rather than monthly, for example.

21 For example, during the period March–April 2015, the Chinese equity index rose, while the UK equity index fell.

22 But have been selected to permit explicit presentation of the analysis methods used.

23 Generally practitioners are more familiar with percentage returns, rather than decimals.

24 The sample standard deviation of the monthly returns, with denominator $(N-1)$ was used, giving the monthly volatility, which was then divided by $\sqrt{\delta t}$, with $\delta t = 1/12$, to obtain annualised volatilities, as a more familiar unit for practitioners.

25 The eigenvalues for this correlation matrix are 0.5899, 0.7430, 1.1764, 1.4907, meaning that this matrix is positive semi-definite, as all of its eigenvalues are positive or zero (ie none of its eigenvalues is negative). This is a necessary property for a valid correlation matrix.

26 Thus a 30 per cent portfolio weight was allocated to each of UK fixed interest, UK equity and US equity, with a lesser 10 per cent allocation to Chinese equity, while the sum of the portfolio weights added up to 100 per cent.

27 The portfolio 95 per cent monthly parametric VaR for the initial sub-period was calculated as $VaR_{\%} = \cdot -N\sigma\delta t^{1/2} \cdot = 1.64485 \times 5.09\% \times \sqrt{1/12} = 2.42\%$. While parametric VaR may be criticised as a risk measure, and indeed a range of other VaR methodologies are available, the primary focus of the current paper is on variance-covariance matrix stress-testing; thus parametric VaR was deemed to be sufficient for the present purpose.

28 Even though monthly data have been used here, this approach can easily be generalised to longer periods and different sampling frequencies, for weekly or daily data, etc.

29 For the correlation matrix, n^2; minus n for each element in the leading diagonal, with the remainder divided by two. A little thought reveals that for an n-asset portfolio there will be $n(n-1)/2$ combinations of asset pairs to be considered.

30 The reader may recall that, in order to use publically available data, the Lyxor ETF iBoxx GBP Gilts were used from *Yahoo Finance*.

31 Or, as monthly data are being used, from 31st December, 2010.

32 We imagine that we are conducting the stress-tests immediately after the end of April 2015, so the final period data until 31st August, 2015 are unavailable for the stress-test calculations.

33 A similar approach could be used to scope a likely range of values for stressed volatilities, perhaps using rolling periods to determine volatility ranges. These could be used in the following stress-test. For the sake of simplicity and brevity, these have not been included in the current study, which aims primarily to explore stress-testing in terms of correlations. The approaches used below are, however, capable of including adjusted volatilities as well as targeted correlations.

34 Indeed the eigenvalues for $\widehat{\boldsymbol{R}}_{Target}$ are -0.2704, 0.9431, 1.2768 and 2.0504, meaning that the matrix is not positive semi-definite and therefore not a valid correlation matrix.

35 That is, will be positive semi-definite, while as nearly as possible retaining the targeted correlation values.

36 Which are then rescaled if original asset variances are to be left unchanged.

37 To parody a quote, 'a rising θ will lift all correlations'. 'A rising tide lifts all boats', improvements in the economy will benefit all participants in that economy, attributed to John F Kennedy in a speech from 1963. See Wikipedia (n.d.) 'A rising tide lifts all boats', available at: https://en.wikipedia.org/wiki/A_rising_tide_lifts_all_boats (accessed 13th October, 2015).

38 Or else to adjust volatilities as part of the stress test.

39 In fact, even without a search algorithm, it is quite straightforward to set up a spreadsheet of calculations [F1, F2] (see Appendix), followed by calculations of the appropriate correlations and their average. Trial values of θ can then be entered manually, until such time as the resulting average correlation has a value of 0.785.

40 Or equivalently between the correlations on the returns pairs R'_{UK}/R'_{US} and R'_{UK}/R'_{CH}.

41 This matrix has eigenvalues 0.1687, 0.2807, 0.9634 and 2.5872; as these are all positive, the matrix is positive semi-definite, meeting the requirement of a valid correlation matrix.

42 This result seems unsurprising, as the target correlations selected required a high correlation between UK equities and US equities as well as between UK equities and CH equities; hence with UK equities acting as a link there must also be a high correlation between US equities and CH equities. This was consistent with the concern raised during the definition of the stress-test scenarios.

43 Generally during a portfolio stress-test it is assumed the portfolio manager has no time to react to some market crisis event, or else the market crisis results in market illiquidity that prevents portfolio asset weights from being adjusted.

44 Rapisarda, F., Brigo, D. and Mercurio, F. (2007) 'Parameterizing correlations: a geometric interpretation', *IMA Journal of Management Mathematics*, Vol. 18, No. 1, pp. 55–73.

45 Which was previously noted not to be positive semi-definite and therefore not a valid correlation matrix.

46 Reordering the assets makes no difference to the eigenvalues of the matrix, which remains

positive semi-definite and a valid correlation matrix. The reordered matrix $\boldsymbol{C}_{Initial}$ remains a valid correlation matrix, as it could easily have been constructed with the assets in that order in the first instance.

47 Which remains not positive semi-definite and therefore not a valid correlation matrix.

48 The interested reader should find that upon multiplying this out with its transpose to numerical rounding they obtain $\boldsymbol{C}_{Initial}$.

49 Substitution of the angles into $\boldsymbol{B}_{4\times4}$ permits the interested reader to recreate $\boldsymbol{U}$.

50 It can be seen that $\boldsymbol{A}$ is the top-left 3×3 elements of $\widehat{\boldsymbol{C}}_{Target}$.

51 In other words the submatrix must also be positive semi-definite.

52 In this case 'acceptable' means that the submatrix $\boldsymbol{A}$ is positive semi-definite (and therefore a valid correlation matrix) and does not generate complex numbers during the Cholesky decomposition.

53 The full range of applicability of such a relationship is not examined here; it is proposed only as a reasonable and practical solution to the current problem.

54 Substitution of these angles into $\boldsymbol{B}_{3\times3}$ recreates $\boldsymbol{V}$.

55 The portfolio asset weights were deemed to be unchanged, and as only the correlation matrix has been adjusted there is no need to be concerned with any potential impact on asset volatilities.

56 It is worth noting that the above calculation of parametric VaR uses the original volatility matrix **v**. If the practitioner wished to use some adjusted volatilities in the stress-test these could be straightforwardly incorporated at this stage.

57 The reader will recall that, as N&B's matrix $\boldsymbol{A}$ was not positive semi-definite and generated a complex number during Cholesky decomposition during the first attempt, the US equity, Chinese equity correlation was modified before proceeding further. During this modification, a choice of two correlation values was possible, with the value nearest to the target being selected. This might be regarded as a potential modest development of N&B's approach.

58 The portfolio 95 per cent monthly parametric VaR for the terminal sub-period was calculated as $VaR_{\%} = \bullet -N\sigma\delta t^{1/2} \bullet = 1.64485 \times 9.30\% \times \sqrt{1/12} = 4.42\%$.

59 Should the reader wish to include adjusted volatilities, the appropriate points at which these could be incorporated into the calculations have been indicated in the sections above.

60 In this case a multiplier of four times volatility was used.

61 Although the stress test scenarios used only explored changes to portfolio correlations, without making any adjustments to volatility.

62 Potentially including rescaling volatilities.

63 As well as level of intellectual comfort.

64 Here US–UK equity and UK–Chinese equity.

APPENDIX A: MODIFICATION OF RETURN VECTORS FOLLOWING FINGER

Adjusting the variance–covariance matrix using the method proposed by Finger involves changing correlations by modifying selected return vectors period by period, which are then rescaled if original asset variances are to be left unchanged.

The application of the method proceeded as follows. Initially consider the 12-monthly returns on the four assets shown in Table 1. As the correlation pairs to be adjusted are UK equity/US equity and UK equity/CH equity, returns from three assets were to be modified; UK equity, US equity and CH equity. The first step was to calculate the average period returns for the three assets identified.

$$R_{ave} = \frac{1}{3}\ (R_{UK} + R_{US} + R_{CH})$$

In the above, R_{ave} is a single-period average return, while R_{UK}, R_{US} and R_{CH} are the returns on UK equity, US equity and CH equity, respectively, for the same period. Modified period returns for UK equity, US equity and CH equity were then constructed as weighted averages of R_{ave} and the relevant asset class using θ (and requiring that $0 \leq \theta \leq 1$) as

$$\begin{aligned} R'_{UK} &= \theta R_{ave} + (1-\theta)\, R_{UK} \\ R'_{US} &= \theta R_{ave} + (1-\theta)\, R_{US} \\ R'_{CH} &= \theta R_{ave} + (1-\theta)\, R_{CH} \end{aligned} \qquad \text{[F1]}$$

From the above, it can be seen that as θ increases towards unity, the period returns on each asset approach R_{ave}, with the consequence that their correlations would also approach unity. The resulting modified period returns are shown in Table 3 using $\theta = 0.6408$, together with R_{ave} (at this stage the reader is asked to accept the value of θ offered, an explanation of how it is obtained follows later).

As a result of applying [F1] to the period returns, the volatilities for the modified returns R'_{UK}, etc differ from those of R_{UK}, etc. Table 3 shows that the annualised volatility for R'_{UK} is $\sigma'_{UK} = 6.04\%$ compared with $\sigma_{UK} = 7.55\%$ for R_{UK}, with differences also occurring for R'_{US} and R'_{CH}. To restore the original asset volatilities, Finger suggests normalising the modified returns:

$$\begin{aligned} R''_{UK} &= R'_{UK} \times \frac{\sigma_{UK}}{\sigma'_{UK}} \\ R''_{US} &= R'_{US} \times \frac{\sigma_{US}}{\sigma'_{US}} \\ R''_{CH} &= R'_{CH} \times \frac{\sigma_{CH}}{\sigma'_{CH}} \end{aligned} \qquad \text{[F2]}$$

The resulting normalised returns appear in Table 3, where it can be seen that the asset volatilities now match those of the original asset data ($\sigma''_{UK} = \sigma_{UK}$, $\sigma''_{US} = \sigma_{US}$ and $\sigma''_{CH} = \sigma_{CH}$). Although not included here, should adjusted volatilities also be desired in the stress-test, this step would be the appropriate place at which to introduce them, so that the modified returns series would then include both adjusted correlations and volatilities.

The value of θ is obtained by completing the calculation [F1, F2] with a trial value (say $\theta = 0.5$). A correlation matrix can then be constructed from either modified returns, R'_{UK}, R'_{US}, R'_{CH}, or normalised returns R''_{UK}, R''_{US}, R''_{CH}, as the normalisation process does not affect the correlations. A search algorithm was then used to explore trial values of θ until the desired correlation target was obtained.

APPENDIX B: OBTAINING CORRELATIVE ANGLES FOLLOWING NUMPACHAROEN AND BUNWONG

Following N&B, one commences by defining matrix $\boldsymbol{B}$, with elements

$$b_{ij} = \begin{cases} \cos\theta_{ij} \cdot \prod_{k=1}^{j-1} \sin\theta_{ik} & \text{for } j = 1 \text{ to } n-1 \\ \prod_{k=1}^{j-1} \sin\theta_{ik} & \text{for } j = n \end{cases}.$$

In the case of a 4×4 matrix this gives:

$$\boldsymbol{B}_{4\times4} = \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12}\sin\theta_{11} & \cos\theta_{13}\sin\theta_{11}\sin\theta_{12} & \sin\theta_{11}\sin\theta_{12}\sin\theta_{13} \\ \cos\theta_{21} & \cos\theta_{22}\sin\theta_{21} & \cos\theta_{23}\sin\theta_{21}\sin\theta_{22} & \sin\theta_{21}\sin\theta_{22}\sin\theta_{23} \\ \cos\theta_{31} & \cos\theta_{32}\sin\theta_{31} & \cos\theta_{33}\sin\theta_{31}\sin\theta_{32} & \sin\theta_{31}\sin\theta_{32}\sin\theta_{33} \\ \cos\theta_{41} & \cos\theta_{42}\sin\theta_{41} & \cos\theta_{43}\sin\theta_{41}\sin\theta_{42} & \sin\theta_{41}\sin\theta_{42}\sin\theta_{43} \end{bmatrix}.$$

By setting diagonal angle $\theta_{ii} = 0$ for all i, $\boldsymbol{B}$ simplifies to

$$\boldsymbol{B}_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \cos\theta_{21} & \sin\theta_{21} & 0 & 0 \\ \cos\theta_{31} & \cos\theta_{32}\sin\theta_{31} & \sin\theta_{31}\sin\theta_{32} & 0 \\ \cos\theta_{41} & \cos\theta_{42}\sin\theta_{41} & \cos\theta_{43}\sin\theta_{41}\sin\theta_{42} & \sin\theta_{41}\sin\theta_{42}\sin\theta_{43} \end{bmatrix}.$$

At this stage, we can note that products of sines and cosines will always take values over the range −1 to +1, making them suitable for representing correlations.

Some correlation matrix $\boldsymbol{C}$ is represented as the product of a lower triangular matrix and its transpose using Cholesky decomposition. Cholesky decomposition is valid for Hermitian positive definite matrices; as a square symmetric matrix with only real entries is Hermitian, it can be applied to any correlation matrix. Numerical algorithms are available to execute the Cholesky decomposition, which would be essential for a portfolio of many assets. By Cholesky decomposition our correlation matrix may be decomposed into a lower triangular matrix and its transpose $\boldsymbol{C}=\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}}$.

The next stage is to set $\boldsymbol{U}=\boldsymbol{B}_{4\times 4}$ above and to solve for the correlative angles:

$$\boldsymbol{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \cos\theta_{21} & \sin\theta_{21} & 0 & 0 \\ \cos\theta_{31} & \cos\theta_{32}\sin\theta_{31} & \sin\theta_{31}\sin\theta_{32} & 0 \\ \cos\theta_{41} & \cos\theta_{42}\sin\theta_{41} & \cos\theta_{43}\sin\theta_{41}\sin\theta_{42} & \sin\theta_{41}\sin\theta_{42}\sin\theta_{43} \end{bmatrix}.$$

This results in a matrix of correlative angles:

$$\theta_{4\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \theta_{21} & 0 & 0 & 0 \\ \theta_{31} & \theta_{32} & 0 & 0 \\ \theta_{41} & \theta_{42} & \theta_{43} & 0 \end{bmatrix}.$$

Changes to correlative angles will always result in trigonometrical functions (and their products) taking values over the range −1 to +1. Thus some matrix of revised correlative angles $\widehat{\boldsymbol{\theta}}$ can be created, resulting in a revised lower triangular matrix $\widehat{\boldsymbol{U}}$. The product of $\widehat{\boldsymbol{U}}$ and its transpose will then yield a Hermitian positive definite matrix, which has the necessary properties for a correlation matrix.

APPENDIX C: EXAMPLE OF NAÏVE APPLICATION OF CHOLESKY DECOMPOSITION TO NON-POSITIVE SEMI-DEFINITE MATRIX

If one were to naively proceed with Cholesky decomposition on the non-positive semi-definite matrix $\boldsymbol{A}$ and to re-express $\boldsymbol{A}=\boldsymbol{V}\boldsymbol{V}^{\boldsymbol{T}}$, one obtains

$$\boldsymbol{V} = \begin{bmatrix} 1 & 0 & 0 \\ 0.89 & 0.4560 & 0 \\ 0.68 & -1.8360 & 1.6832i \end{bmatrix}.$$

The Cholesky decomposition has inconveniently generated a complex number $\left(i=\sqrt{-1}\right)$ in element V_{33}. Rather than glossing over this (for example, by conveniently selecting less challenging target correlations) the main text explores what a practitioner might do when faced with this sort of situation arising from real-world data.

The practitioner might, however, worry that there is an error in the Cholesky decomposition, but this is not the case. Due to the lower diagonal structure of $\boldsymbol{V}$, most of the elements of $\boldsymbol{V}\boldsymbol{V}^{\boldsymbol{T}}$ that involve V_{33} are multiplied by zero. The only one that is not is the bottom-right '33' element, which is calculated as

$$V_{31}(V^T)_{13} + V_{32}(V^T)_{23} + V_{33}(V^T)_{33} = 0.68^2 + (-1.83597)^2 + (1.68321i)^2 = 1.$$

Conveniently, the only appearance of i is squared when reconstructing matrix $\boldsymbol{A}$.

More serious issues remain, however; N&B's method requires that $\boldsymbol{A}$ is a valid correlation matrix, which as it stands it is not, as it is not positive semi-definite. Further, the next stage would be to express matrix $\boldsymbol{V}$ in terms of correlative angles using trigonometrical functions — a step one would not wish to take when complex numbers are involved.[1]

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APPENDIX D: EXPLORATION OF AN EXPRESSION FOR LINKED CORRELATIONS

Consider a problem with three assets, denoted 'asset 1', 'asset 2' and 'asset 3', having correlations between them ρ_{12}, ρ_{23}, ρ_{13}, where the subscripts denote the assets. Let us say that we have the correlations between assets 1 and 2 (ρ_{12}), and assets 1 and 3 (ρ_{13}) that we wish to target, and desire to determine a 'sensible' estimate for the correlation between assets 2 and 3 (ρ_{23}). By 'sensible' we note that we are motivated by the example in the main text, and desire a value of ρ_{23} that will ensure that the resulting matrix is positive semi-definite (and therefore a valid correlation matrix) and that we avoid obtaining a complex value for element '33' during a Cholesky decomposition of a 3×3 correlation matrix based on the above correlations.

Determination of a suitable expression linking the correlations $\rho_{23} = f(\rho_{12}, \rho_{13})$ can be derived either by consideration of the eigenvalues, or else the conditions necessary within the Cholesky decomposition in order to avoid generation of a complex number. A geometrical interpretation of the resulting expression is also offered.

Eigenvalues

Define 3×3 correlation matrix

$$\boldsymbol{C}_{\mathbf{3\times3}} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}.$$

This will have eigenvalues, λ satisfying

$$det(\boldsymbol{C}_{\mathbf{3\times3}} - \lambda \boldsymbol{I}) = 0,$$

Here $\boldsymbol{I}$ is the identity matrix and 'det' signifies taking the determinant. Thus,

$$det(\boldsymbol{C}_{\mathbf{3\times3}} - \lambda \boldsymbol{I}) = det \begin{bmatrix} 1-\lambda & \rho_{12} & \rho_{13} \\ \rho_{12} & 1-\lambda & \rho_{23} \\ \rho_{13} & \rho_{23} & 1-\lambda \end{bmatrix} = 0.$$

Multiplying this out gives:

$$\rho_{13}\big(\rho_{12}\rho_{23} - \rho_{13}(1-\lambda)\big) + \rho_{23}\big(\rho_{12}\rho_{13} - \rho_{23}(1-\lambda)\big) + (1-\lambda)\big((1-\lambda)^2 - {\rho_{12}}^2\big) = 0,$$

$$2\rho_{12}\rho_{13}\rho_{23} + (1-\lambda)^3 - (1-\lambda){\rho_{12}}^2 - (1-\lambda)\rho_{13}^2 - (1-\lambda)\rho_{23}^2 = 0.$$

Collect terms in ρ_{23} to get

$$-(1-\lambda)\rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23} + (1-\lambda)\left((1-\lambda)^2 - {\rho_{12}}^2 - \rho_{13}^2\right) = 0.$$

This quadratic in ρ_{23} has solutions

$$\rho_{23} = \frac{\rho_{12}\rho_{13} \mp \sqrt{\rho_{12}^2\rho_{13}^2 + (1-\lambda)^2\big((1-\lambda)^2 - {\rho_{12}}^2 - \rho_{13}^2\big)}}{(1-\lambda)}.$$

Now seek a value of $\rho_{23} = f(\rho_{12,}\rho_{13})$ with a zero eigenvalue, thus $\lambda = 0$, and

$$\rho_{23} = \rho_{12}\rho_{13} \mp \sqrt{\rho_{12}^2\rho_{13}^2 + 1 - {\rho_{12}}^2 - \rho_{13}^2}.$$

Factorising the term inside the square-root gives:

$$\rho_{23} = \rho_{12}\rho_{13} \mp \sqrt{(1-{\rho_{12}}^2)} \times \sqrt{(1-{\rho_{13}}^2)}. \qquad \text{[D1]}$$

For the example used in the main text, let us say asset 1 is UK equity, asset 2 is US equity and asset 3 is Chinese equity; then $\rho_{12} = 0.89$ (UK–US), $\rho_{13} = 0.68$ (UK–China) and we desire an estimate for ρ_{23} (US–China) that will result in a zero eigenvalue. Thus

$$\rho_{23} = 0.68 \times 0.89 \pm \sqrt{1-0.68^2} \times \sqrt{1-0.89^2}$$
$$= 0.6052 \pm \sqrt{0.1118} = 0.2709 \text{ or } 0.9395.$$

From a mathematical perspective, either of the two values above would generate a zero eigenvalue; however, as mentioned in the main text, the value $\rho_{23} = 0.2709$ lies closer to the initial correlation matrix and, as it was not desired specifically to target ρ_{23} during the stress-test, it would appear reasonable to select the solution closer to the initial correlation value between US and Chinese equities of −0.23.

As indicated in the main text, the resulting matrix is

$$\boldsymbol{C}_{\mathbf{3\times3}} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.89 & 0.68 \\ 0.89 & 1 & 0.2709 \\ 0.68 & 0.2709 & 1 \end{bmatrix}.$$

This has eigenvalues of 0, 0.7398 and 2.260, meaning that it is positive semi-definite and a valid correlation matrix.

Cholesky decomposition

Define 3×3 correlation matrix, as before:

$$C_{3\times3} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}.$$

As $\boldsymbol{C}$ is square symmetric with real elements it is Hermitian and so can be expressed via Cholesky decomposition as the product of a lower triangular matrix and its transpose.

$$\boldsymbol{C} = \boldsymbol{U}\boldsymbol{U}^{\boldsymbol{T}} = \begin{bmatrix} U_{11} & 0 & 0 \\ U_{21} & U_{22} & 0 \\ U_{31} & U_{32} & U_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{21} & U_{31} \\ 0 & U_{22} & U_{32} \\ 0 & 0 & U_{33} \end{bmatrix}.$$

For given ρ_{12} and ρ_{23}, we wish to explore conditions to ensure that U_{33} is a real number. Thus:

$$\begin{bmatrix} U_{11}^2 & U_{21}U_{11} & U_{31}U_{11} \\ U_{21}U_{11} & U_{21}^2 + U_{22}^2 & U_{31}U_{21} + U_{32}U_{22} \\ U_{31}U_{11} & U_{31}U_{21} + U_{32}U_{22} & U_{31}^2 + U_{32}^2 + U_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}.$$

For any correlation matrix the leading diagonal elements have value unity, so $U_{11}=1$, $U_{21}=\rho_{12}$ and $U_{31}=\rho_{13}$. Also for C_{33},

$$U_{31}^2 + U_{32}^2 + U_{33}^2 = 1.$$

For U_{33} to be real, require $U_{33}^2 \geq 0$, to avoid taking the square root of a negative value.

$$U_{33}^2 = 1 - U_{31}^2 - U_{32}^2 = 1 - \rho_{13}^2 - U_{32}^2 \geq 0$$

Proceeding

$$1 - \rho_{13}^2 - U_{32}^2 \geq 0,$$

$$\rho_{13}^2 + U_{32}^2 \leq 1,$$

$$U_{32} \leq \pm\sqrt{1-\rho_{13}^2}.$$

To link the above condition on U_{32} to ρ_{23} we use $U_{31}U_{21} + U_{32}U_{22} = \rho_{23}$ as follows:

$$U_{32} = \frac{\rho_{23} - U_{31}U_{21}}{U_{22}},$$

$$U_{32} = \frac{\rho_{23} - \rho_{13}\rho_{12}}{U_{22}}.$$

Now require an expression for U_{22}, which we obtain from $U_{21}^2 + U_{22}^2 = 1$, as

$$U_{22}^2 = 1 - U_{21}^2,$$

$$U_{22} = \pm\sqrt{1-\rho_{12}^2}.$$

Considering only the boundary of the inequality

$$U_{32} = \pm\sqrt{1-\rho_{13}^2},$$

$$\frac{\rho_{23} - \rho_{13}\rho_{12}}{\pm\sqrt{1-\rho_{12}^2}} = \pm\sqrt{1-\rho_{13}^2},$$

$$\rho_{23} = \rho_{13}\rho_{12} \pm \sqrt{1-\rho_{13}^2} \times \sqrt{1-\rho_{12}^2}. \qquad \text{[D2]}$$

This is the identical result to [D1], although in this case it has been demonstrated that the resulting value of ρ_{23} will not result in a complex value of U_{33} in the Cholesky decomposition. From a mathematical perspective, either of the two values above generates real U_{33}; however, as before, the value ρ_{23}=0.2709 lies closer to the initial correlation matrix.

Geometrical interpretation

A geometrical interpretation of [D1] and [D2] is also possible.[2]

Consider correlation vectors:

$$\vec{a} = \begin{pmatrix} \rho_{12} \\ \sqrt{1-\rho_{12}^2} \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} \rho_{13} \\ \sqrt{1-\rho_{13}^2} \end{pmatrix}.$$

These can be interpreted as shown in Figure D1, being vectors of unit length making angles θ_{12} and θ_{13}, respectively, with the x-axis and having projections ρ_{12} and ρ_{13} onto the x-axis. We note that by Pythagoras, the projections onto the y-axis can be obtained from

$$\rho_{12}^2 + (\Delta y)^2 = 1^2,$$

$$\Delta y = \sqrt{1-\rho_{12}^2}.$$

With a similar calculation for ρ_{13}. Thus we obtain both x and y-components of the vectors $\vec{a}$ and $\vec{b}$. We see that these vectors are just representations in the (x, y) plane of the correlation as unit length vectors around a circle.

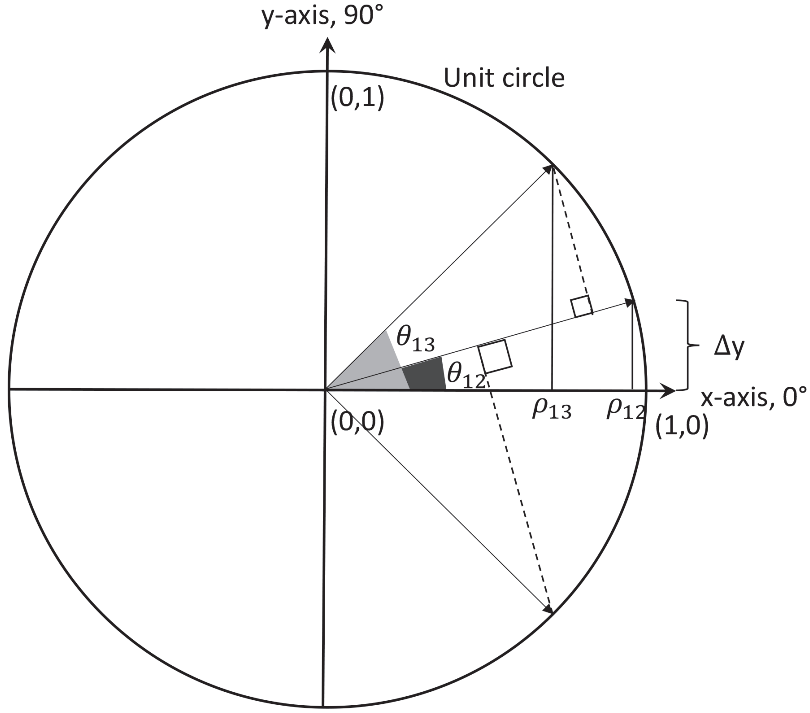

Figure D1: Correlation values and angles. The correlation values ρ_{12}, ρ_{13} are the projections of the vectors onto the *x*-axis

The projection of $\vec{b}$ onto $\vec{a}$ is given by the scalar (dot) product. We also consider the possibility of reflection of one or other vector in the *x*-axis so that its *y*-coordinate is negative by using both the positive and negative roots:

$$\vec{a} = \begin{pmatrix} \rho_{12} \\ \pm\sqrt{1-\rho_{12}^2} \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} \rho_{13} \\ \pm\sqrt{1-\rho_{13}^2} \end{pmatrix}.$$

The expression for the scalar product of the above is

$$\vec{a} \cdot \vec{b} = \rho_{12}\rho_{13} \pm \sqrt{1-\rho_{12}^2} \times \sqrt{1-\rho_{13}^2}. \qquad \text{[D3]}$$

This is identical to [D1] and [D2], demonstrating that the expression for ρ_{23} is geometrically the projection of correlation vector $\vec{b}$ onto $\vec{a}$. The reflection in the *x*-axis permits two roots to the expression.

As the correlation vectors $\vec{a}$ and $\vec{b}$ are both of unit length, the degree of correlation is captured by the angle between the vectors. If the vectors are orthogonal there is no correlation between them; similarly with zero angle between them they are perfectly correlated.

Another expression for the scalar product is $\vec{a} \cdot \vec{b} = ab\cos\theta$, where θ is the angle between the correlation vectors. These angles (θ_{12} and θ_{13}) can be readily explored using trigonometry.

$$\tan\theta_{12} = \frac{\pm\sqrt{1-\rho_{12}^2}}{\rho_{12}} \quad \text{and} \quad \tan\theta_{13} = \frac{\pm\sqrt{1-\rho_{13}^2}}{\rho_{13}}.$$

The '±' indicates the possibility of reflection in the *x*-axis. Considering the projection of vector $\vec{b}$ onto $\vec{a}$, we know that the scalar lengths of each of the vectors is unity. Combining [D1] and [D2]:

$$\rho_{23} = \vec{a} \cdot \vec{b} = 1 \times 1 \times \cos\theta = \cos\theta.$$

Recalling that θ is the angle between θ_{12} and θ_{13}. Applying this to the example in the main text with $\rho_{12}=0.89$ (UK–US), and $\rho_{13}=0.68$ (UK–China), we obtain

$$\tan\theta_{12} = \pm 0.51232 \quad \text{and} \quad \tan\theta_{13} = \pm 1.078253,$$

$$\theta_{12} = \pm 0.47345 \ \textit{radians} \quad \text{and} \quad \theta_{13} = \pm 0.82303 \ \textit{radians}.$$

The differences between the angles can be either

$$\theta = 0.82303 - 0.47345 = 0.34958 \ \textit{radians},$$

or

$$\theta = 0.82303 - (-0.47345) = 1.29648 \ \textit{radians}.$$

This gives $\cos\theta = 0.9395$ *or* 0.2709, the same result as before. In degrees these angles evaluate at

$$\theta_{12} = \frac{0.47345}{2\pi} \times 360° = 27.13°,$$

$$\theta_{13} = \frac{0.82303}{2\pi} \times 360° = 47.16°,$$

$$\theta = \frac{0.34958}{2\pi} \times 360° = 20.03°, \text{ and}$$

$$\theta = \frac{1.29648}{2\pi} \times 360° = 74.28°.$$

Which agree to rounding accuracy.

Appendix references

1 Actually expressions based on Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$ can be used to proceed with complex angles; however, this is not explored in the current paper.

2 An extensive and more theoretical discussion of the geometrical interpretation of correlations is given in Rapisarda F., Brigo D. and Mercurio F. (2007) 'Parameterizing correlations: a geometric interpretation', *IMA Journal of Management Mathematics*, Vol. 18, No. 1, pp. 55–73.

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